ω -MODELS OF FINITE SET THEORY

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ABSTRACT. Finite set theory, here denoted $\mathbf{ZF}_{\text{fin}}$, is the theory obtained by replacing the axiom of infinity by its negation in the usual axiomatization of $\mathbf{ZF}$ (Zermelo-Fraenkel set theory). An ω -model of $\mathbf{ZF}_{\text{fin}}$ is a model in which every set has at most finitely many elements (as viewed externally). Mancini and Zambella (2001) employed the Bernays-Rieger method of permutations to construct a recursive ω -model of $\mathbf{ZF}_{\text{fin}}$ that is nonstandard (i.e., not isomorphic to the hereditarily finite sets V_ω). In this paper we initiate the metamathematical investigation of ω -models of $\mathbf{ZF}_{\text{fin}}$. In particular, we present a perspicuous method for constructing recursive nonstandard ω -model of $\mathbf{ZF}_{\text{fin}}$ without the use of permutations. We then use this method to establish the following central theorem.

Theorem A. *For every simple graph (A, F) , where F is a set of unordered pairs of A , there is an ω -model $\mathfrak{M}$ of $\mathbf{ZF}_{\text{fin}}$ whose universe contains A and which satisfies the following two conditions:*

- (1) *There is parameter-free formula $\varphi(x, y)$ such that for all elements a and b of $\mathfrak{M}$, $\mathfrak{M} \models \varphi(a, b)$ iff $\{a, b\} \in F$;*
- (2) *Every element of $\mathfrak{M}$ is definable in $(\mathfrak{M}, c)_{c \in A}$.*

Theorem A enables us to build a variety of ω -models with special features, in particular:

Corollary 1. *Every group can be realized as the automorphism group of an ω -model of $\mathbf{ZF}_{\text{fin}}$.*

Corollary 2. *For each infinite cardinal κ there are 2^κ rigid non-isomorphic ω -models of $\mathbf{ZF}_{\text{fin}}$ of cardinality κ .*

Corollary 3. *There are continuum-many nonisomorphic point-wise definable ω -models of $\mathbf{ZF}_{\text{fin}}$.*

1. INTRODUCTION

In 1953, Kreisel [Kr] and Mostowski [Mo] independently showed that certain finitely axiomatizable systems of set theory formulated in an *expansion* of the usual language $\{\in\}$ of set theory do not possess any recursive models. This result was improved in 1958 by Rabin [Ra] who found a “familiar” finitely axiomatizable first order theory that has no recursive model: Gödel-Bernays¹ set theory **GB** without the axiom of infinity (note that **GB** can be formulated in the language $\{\in\}$ with no extra symbols). These discoveries were overshadowed by Tennenbaum’s celebrated 1961 theorem that characterizes the standard model of **PA** (Peano arithmetic) as the only recursive model of **PA** up to isomorphism, thereby shifting the focus of the investigation of the complexity of models from set theory to arithmetic. We have come a long way since Tennenbaum’s pioneering work in understanding the contours of the “Tennenbaum boundary” that separates those fragments of **PA** that can have a recursive model (such as *IOpen*) from those which do not (such as *I $\exists_1$*), but the study of the complexity of models of arithmetic and fragments remains a vibrant research area with many intriguing open questions.²

The point of departure for the work presented here is Mancini and Zambella’s 2001 paper [MZ] that focuses on *Tennenbaum phenomena in set theory*. Mancini and Zambella introduced a weak fragment (dubbed $\mathbf{KP}\Sigma_1$ ³) of Kripke-Platek set theory **KP**, and showed that the only recursive model of $\mathbf{KP}\Sigma_1$ up to isomorphism is the standard one, i.e., $(V_\omega, \in)$, where V_ω is the set of hereditarily finite sets. In contrast, they used the *Bernays-Rieger*⁴ permutation method to show that the theory

¹We have followed Mostowski’s lead in our adoption of the appellation **GB**, but some set theory texts refer to this theory as **BG**. To make matters more confusing, the same theory is also known in the literature as **VNB** (*von Neumann-Bernays*) and **NBG** (*von Neumann-Bernays-Gödel*).

²See, e.g., the papers by Kaye [Ka] and Schmerl [S-2] in this volume, and Mohsenipour’s [M].

³The axioms of $\mathbf{KP}\Sigma_1$ consist of Extensionality, Pairs, Union, Foundation, Δ_0 -Comprehension, Δ_0 -Collection, and the scheme of $\in$ -Induction (defined in part (e) of Remark 2.2) only for Σ_1 formulas. Note that $\mathbf{KP}\Sigma_1$ does not include the axiom of infinity.

⁴In [MZ] this method is incorrectly referred to as the *Fraenkel-Mostowski* permutation method, but this method was invented by Bernays (announced in [B-1, p. 9], and presented in [B-2]) and fine-tuned by Rieger [Ri] in order to build models of set theory that violate the regularity (foundation) axiom, e.g., by containing sets x such that $x = \{x\}$. In contrast, the Fraenkel-Mostowski method is used to construct models of set theory with atoms in which the axiom of choice fails.

$\mathbf{ZF}_{\text{fin}}$ obtained by replacing the axiom of infinity by its negation in the usual axiomatization of $\mathbf{ZF}$ (Zermelo-Fraenkel set theory) has a recursive nonstandard model. The Mancini-Zambella recursive nonstandard model also has the curious feature of being an ω -*model* in the sense that every element of the model, as viewed externally, has at most finitely many members, a feature that caught our imagination and prompted us to initiate the systematic investigation of the metamathematics of ω -models of $\mathbf{ZF}_{\text{fin}}$.

The plan of the paper is as follows. After going through preliminaries, in Section 3 we present a simple robust construction of ω -models of $\mathbf{ZF}_{\text{fin}}$, a construction that leads to a perspicuous proof of the existence of recursive nonstandard ω -model of $\mathbf{ZF}_{\text{fin}}$ without the use of permutations. In Section 4 we fine-tune the method of Section 3 to show the existence of a wealth of nonisomorphic ω -models of $\mathbf{ZF}_{\text{fin}}$. The central theorem of Section 4 is Theorem 4.2 which shows that every simple graph can be canonically coded into an ω -model $\mathfrak{M}_G$ of $\mathbf{ZF}_{\text{fin}}$. This theorem enables us to build a variety of ω -models of $\mathbf{ZF}_{\text{fin}}$ with special features. In particular, in Corollary 4.3 we show that every group can be realized as the automorphism group of an ω -model of $\mathbf{ZF}_{\text{fin}}$; and in Corollary 4.6 we construct continuum-many nonisomorphic pointwise definable ω -models of $\mathbf{ZF}_{\text{fin}}$. We also show that there are continuum-many completions of $\mathbf{ZF}_{\text{fin}}$ that possess a unique ω -model up to isomorphism (Theorem 4.8). We conclude the paper with Section 5, in which we present some open problems.

2. PRELIMINARIES

In this section we give some key definitions and state some basic facts.

Definition 2.1.

(a) Models of set theory are *directed graphs* (hereafter: *digraphs*), i.e., structures of the form $\mathfrak{M} = (M, E)$, where E is a binary relation on M that interprets $\in$. We often write xEy as a shorthand for $\langle x, y \rangle \in E$. For $c \in M$, c_E is the set of “elements” of c , i.e.,

$$c_E := \{m \in M : mEc\}.$$

$\mathfrak{M}$ is *nonstandard* if E is not well-founded, i.e., if there is a sequence $\langle c_n : n \in \omega \rangle$ of elements of M such that $c_{n+1}Ec_n$ for all $n \in \omega$.

(b) We adopt the terminology of Baratella and Ferro [BF] of using **EST** (elementary set theory) to refer to the following theory of sets

EST := Extensionality + Pairs + Union + Replacement.

(c) The theory $\mathbf{ZF}_{\text{fin}}$ is obtained by replacing the axiom of infinity by its negation in the usual axiomatization of $\mathbf{ZF}$ (Zermelo-Fraenkel set theory). More explicitly:

$$\mathbf{ZF}_{\text{fin}} := \text{EST} + \text{Power set} + \text{Regularity}^5 + \neg \text{Infinity},$$

where **Infinity** is the usual axiom of infinity, i.e.,

$$\text{Infinity} := \exists x (\emptyset \in x \wedge \forall y (y \in x \rightarrow y \cup \{y\} \in x)).$$

(d) $\text{Tran}(x)$ is the first order formula that expresses the statement “ x is transitive”, i.e.,

$$\text{Tran}(x) := \forall y \forall z (z \in y \in x \rightarrow z \in x).$$

(e) $\text{TC}(x)$ is the first order formula that expresses the statement “the transitive closure of x is a set”, i.e.,

$$\text{TC}(x) := \exists y (x \subseteq y \wedge \text{Tran}(y)).$$

Overtly, the above formula just says that some superset of x is transitive, but it is easy to see that $\text{TC}(x)$ is equivalent within **EST** to the following statement expressing “there is a smallest transitive set that contains x ”

$$\exists y (x \subseteq y \wedge \text{Tran}(y) \wedge \forall z ((x \subseteq z \wedge \text{Tran}(z)) \rightarrow y \subseteq z)).$$

(f) **TC** denotes the *transitive closure* axiom

$$\text{TC} := \forall x \text{TC}(x).$$

(g) $\mathbb{N}(x)$ [read as “ x is a natural number”] is the formula

$$\text{Ord}(x) \wedge \forall y \in x^+ (y \neq \emptyset \rightarrow \exists z (\text{Ord}(z) \wedge y = z^+)),$$

where $\text{Ord}(x)$ expresses “ x is a (von Neumann) ordinal”, i.e., “ x is a transitive set that is well-ordered by $\in$ ”, and $x^+ := x \cup \{x\}$. It is not hard to see that with this interpretation, the induction scheme is provable within **EST** (see [BF, Sec. 1]), thereby allowing **EST** to interpret **PA**.

(h) For a model $\mathfrak{M} \models \text{EST}$, and $x \in M$, we say that x is *N-finite* if there is a bijection in $\mathfrak{M}$ between x and some element of $\mathbb{N}$. Let:

$$(V_\omega)^\mathfrak{M} := \{m \in M : \mathfrak{M} \models \text{“TC}(m) \text{ and } m \text{ is } N\text{-finite”}\}$$

It is easy to see that

$$(V_\omega)^\mathfrak{M} \models \mathbf{ZF}_{\text{fin}} + \text{TC}.$$

⁵The regularity axiom is also known as the *foundation* axiom, stating that every nonempty set has an $\in$ -minimal element.

This shows that the theory $\mathbf{ZF}_{\text{fin}} + \mathbf{TC}$ is interpretable within $\mathbf{EST}$. On the other hand, the Mancini-Zambella proof [MZ, Theorem 3.1], or the proof presented in the next section can be used to show that, conversely, $\mathbf{ZF}_{\text{fin}} + \neg \mathbf{TC}$ is also interpretable in $\mathbf{EST}$. One can show that the above interpretation of $\mathbf{ZF}_{\text{fin}} + \mathbf{TC}$ within $\mathbf{EST}$ is *faithful* (i.e., the sentences that are provably true in the interpretation are precisely the logical consequences of $\mathbf{ZF}_{\text{fin}} + \mathbf{TC}$), but the Mancini-Zambella interpretation of $\mathbf{ZF}_{\text{fin}} + \neg \mathbf{TC}$ within $\mathbf{EST}$ is not faithful⁶.

(i) $\tau(n, x)$ is the term expressing “the n -th approximation to the transitive closure of $\{x\}$ (where n is a natural number)”. Informally speaking,

- $\tau(0, x) = \{x\}$
- $\tau(n+1, x) = \tau(n, x) \cup \{y : \exists z (y \in z \in \tau(n, x))\}.$

Thanks to the coding apparatus⁷ of $\mathbf{EST}$ and the provability of the induction scheme within $\mathbf{EST}$ (mentioned earlier in part (g)), the above informal recursion can be formalized within $\mathbf{EST}$ to show that

$$\mathbf{EST} \vdash \forall n \forall x (\mathbb{N}(n) \rightarrow \exists! y (\tau(n, x) = y)).$$

This leads to the following important observation:

(j) Even though the transitive closure of a set need not form a set in $\mathbf{EST}$ (or even in $\mathbf{ZF}_{\text{fin}}$), the transitive closure $\tau(c)$ of a given set c of a model $\mathfrak{M}$ is *first order definable* via:

$$\tau^{\mathfrak{M}}(c) := \{m \in M : \mathfrak{M} \models \exists n (\mathbb{N}(n) \wedge m \in \tau(n, c))\}.$$

This shows that, in the worst case scenario, transitive closures behave like proper classes. Note that if the set $\mathbb{N}^{\mathfrak{M}}$ of natural numbers of $\mathfrak{M}$ contains nonstandard elements, then the external transitive closure of an element c of $\mathfrak{M}$ might be a proper subset of the $\mathfrak{M}$ -transitive closure $\tau^{\mathfrak{M}}(c)$ of c . However, if $\mathbb{N}^{\mathfrak{M}}$ has no nonstandard elements, then the $\mathfrak{M}$ -transitive closure of c coincides with the external transitive closure of c .

⁶Because the interpretation provably satisfies the sentence “the universe can be built from the transitive closure of an element whose transitive closure forms an ω^* -chain”, a sentence that is not a theorem of $\mathbf{ZF}_{\text{fin}} + \neg \mathbf{TC}$. See the paragraph after Corollary 4.7 for more detail.

⁷More specifically: the function $\beta(x, y)$ defined via $\beta(x, y) = z$ iff $\langle x, z \rangle \in y$, conveniently serves as a “ β -function” for coding sequences (where $\langle x, y \rangle$ is the usual Kuratowski ordered pair of x and y).

Remark 2.2.

(a) Vopěnka [Vo-2] has shown that $\mathbf{ZF}_{\text{fin}}$ is provable from the fragment of **AST** (Alternative Set Theory), dubbed **VF** in [BF] whose axioms consist of **Extensionality**, **Empty set**, **Adjunction** (given sets x and y , $x \cup \{y\}$ exists), and the schema of **Set-induction**, consisting of the universal closure of formulas of the form

$$(\theta(0) \wedge \forall x \forall y (\theta(x) \rightarrow \theta(x \cup \{y\}))) \rightarrow \forall x \theta(x).$$

(b) Besides $\neg \mathbf{Infinity}$ there are at least two other noteworthy first order statements that can be used to express “every set is finite”:

- $\mathbf{Fin}_N$: Every set is N -finite.
- $\mathbf{Fin}_D$: Every set is Dedekind-finite (hereafter: D -finite), i.e., no set is equinumerous to a proper subset of itself.

It is easy to see that **EST** proves $\mathbf{Fin}_N \rightarrow \mathbf{Fin}_D \rightarrow \neg \mathbf{Infinity}$. By a theorem of Vopěnka [Vo-1], **Power set** and the well-ordering theorem (and therefore the axiom of choice) are provable within **EST** + $\mathbf{Fin}_N$ (see [BF, Theorem 5] for an exposition).

(c) Kunen [BF, Sec. 7] has shown the consistency of the theory **EST** + $\neg \mathbf{Infinity}$ + $\neg \mathbf{Fin}_N$ using the Fraenkel-Mostowski permutations method. An inspection of Kunen’s proof reveals that indeed $\neg \mathbf{Fin}_D$ also holds in his model. In this light, it would be interesting to see whether one can build a model of **EST** + $\mathbf{Fin}_D$ + $\neg \mathbf{Fin}_N$.

(d) In contrast with Kunen’s aforementioned result, $\mathbf{Fin}_N$ is provable within $\mathbf{ZF}_{\text{fin}} \setminus \{\mathbf{Regularity}\}$ (i.e., within **EST** + **Powerset** + $\neg \mathbf{Infinity}$). To see this, work in $\mathbf{ZF}_{\text{fin}} \setminus \{\mathbf{Regularity}\}$ and suppose to the contrary that there is an element x that is not N -finite. By an easy induction, we find that for every natural number n there is a subset y of x that is equinumerous with n . Now we define the function F on the powerset of x by:

$$F(y) := n, \text{ if } n \text{ is a natural number that is equinumerous with } y; \\ \text{otherwise } F(y) = 0.$$

It is easy to see that the range of F is ω . Hence by **Replacement**, ω is a set. Quod non.

(e) As observed by Kaye and Wong [KW, Proposition 12] within **EST**, the principle **TC** is equivalent to the scheme of **$\in$ -Induction** consisting of statements of the following form (θ is allowed to have suppressed parameters)

$$\forall y (\forall x \in y \theta(x) \rightarrow \theta(y)) \rightarrow \forall z \theta(z).$$

In the same paper Kaye and Wong corrected the commonly held misconception that PA and ZF_{fin} are bi-interpretable (in the sense of [H], or [Vi]) by showing the bi-interpretability of PA and $\text{ZF}_{\text{fin}} + \text{TC}$. In particular, they showed that TC holds in the Ackermann interpretation of ZF_{fin} within PA .

3. BUILDING ω -MODELS

Definition 3.1. Suppose $\mathfrak{M}$ is a model of EST . $\mathfrak{M}$ is an ω -model if $|x_E|$ is finite for every $x \in M$ satisfying $\mathfrak{M} \models “x \text{ is } N\text{-finite}”$ ⁸. It is easy to see that $\mathfrak{M}$ is an ω -model iff $(\mathbb{N}, \in)^{\mathfrak{M}}$ is well-founded.

The following proposition provides a useful *graph-theoretic* characterization of ω -models of ZF_{fin} . Note that even though ZF_{fin} is not finitely axiomatizable⁹, the equivalence of (a) and (b) shows that there is a single sentence in the language of set theory whose well-founded models are precisely the class of ω -models of ZF_{fin} .

- Recall that a vertex x of a digraph $G := (X, E)$ has *finite in-degree* if x_E is finite; and G is *acyclic* if there is no finite sequence $x_1 E x_2 \cdots E x_{n-1} E x_n$ in G with $x_1 = x_n$.

Proposition 3.2. *The following three conditions are equivalent for a digraph $G := (X, E)$:*

- (a) G is an ω -model of ZF_{fin} .
- (b) G is an ω -model of Extensionality, Empty set, Regularity, Adjunction and $\neg$ Infinity.
- (c) G satisfies the following four conditions:
 - (i) E is extensional;
 - (ii) Every vertex of G has finite in-degree;
 - (iii) G is acyclic; and

⁸Note that there are really two salient notions of ω -model, to wit the notion we defined here could be called ω_N -model, and the notion which is defined using ‘ D -finite’ instead of ‘ N -finite’ may be called ω_D -model. For our study of ZF_{fin} the choice is immaterial, since ZF_{fin} proves that every set is N -finite (see part (d) of Remark 2.2).

⁹By a general result of Proof Theory [HP, Cor. 3.24, Chap. III], if T is finitely axiomatizable sequential theory (i.e., a β -function for coding sequences is available in the definitional extension of T), and T interprets a modicum of arithmetic, then T cannot prove all instances of the induction scheme. Therefore, ZF_{fin} is not finitely axiomatizable (see part (g) of Definition 2.1).

(iv) G has an element of in-degree 0, and for all positive $n \in \omega$,

$$(X, E) \models \forall x_1 \cdots \forall x_n \exists y \forall z (zEy \leftrightarrow \bigvee_{i=1}^n z = x_i).$$

Proof:

(a) $\Rightarrow$ (b) : Trivial.

(b) $\Rightarrow$ (c) : Assuming (b), (i) and (ii) are trivially true. To verify (iii), suppose to the contrary that $x_1 E x_2 \cdots E x_{n-1} E x_n$ is a cycle in G with $x_1 = x_n$. By n -applications of **Adjunction**, there is an element $y \in X$ with $y_E = \{x_i : 1 \leq i \leq n\}$. This contradicts **Regularity** since such a y has no minimal “element”. (iv) is an easy consequence of **Empty set** and **Adjunction**.

(c) $\Rightarrow$ (a) : Routine, but we briefly comment on the verification of **Regularity**, which is accomplished by contradiction: if x is a nonempty set with no minimal element, then there exists an external infinite sequence $\langle x_n : n \in \omega \rangle$ of elements of c such that $x_{n+1} E x_n$ for all $n \in \omega$. Invoking statement (ii) of (c), this shows that there are x_m and x_n with $m < n$ such that $x_m = x_n$, which contradicts condition (iii) of (c). $\square$

The following list of definitions prepares the way for the central theorem of this section (Theorem 3.4), which will enable us to build plenty of ω -models of $\mathbf{ZF}_{\text{fin}}$.

Definition 3.3. Suppose $G := (X, E)$ is an extensional, acyclic digraph, all of whose vertices have finite in-degree.

(a) A subset S of X is said to be *coded* in G , if there is some $x \in X$ such that $S = x_E$.

(b) $D(G) := \{S \subseteq X : S \text{ is finite and } S \text{ is not coded in } G\}$. We shall refer to $D(G)$ as the *deficiency set* of G .

(c) Without loss of generality, we assume that for *all* considered digraphs $G = (X, E)$, $X \cap D(G) = \emptyset$.

(d) The infinite sequence of digraphs

$$\langle \mathbb{V}_n(G) : n \in \omega \rangle,$$

where $\mathbb{V}_n(G) := (V_n(G), E_n(G))$, is built recursively using the following clauses:

- $V_0(G) := X$; $E_0(G) := E$.
- $V_{n+1}(G) := V_n(G) \cup D(\mathbb{V}_n(G))$;
- $E_{n+1}(G) := E_n(G) \cup \{\langle x, X \rangle \in V_n(G) \times D(\mathbb{V}_n(G)) : x \in X\}$.

(e) $\mathbb{V}_\omega(G) := (V_\omega(G), E_\omega(G))$, where

$$V_\omega(G) := \bigcup_{n \in \omega} V_n(G), \quad E_\omega(G) := \bigcup_{n \in \omega} E_n(G).$$

Theorem 3.4. *If $G := (X, E)$ is extensional, acyclic digraph, all of whose vertices have finite in-degree, then $\mathbb{V}_\omega(G)$ is an ω -model of $\mathbf{ZF}_{\text{fin}}$.*

Proof: We shall show that $\mathbb{V}_\omega(G)$ satisfies the four conditions of (c) of Proposition 3.2. Before doing so, let us make an observation that is helpful for the proof, whose verification is left to the reader (part (c) of Definition 3.3 comes handy here).

Observation 3.4.1. *For each $n \in \omega$, $\mathbb{V}_{n+1}(G)$ “end extends” $\mathbb{V}_n(G)$, i.e., if $a E_{n+1} b$ and $b \in V_n$, then $a \in V_n(G)$.*

It is easy to see that extensionality is preserved in the passage from $\mathbb{V}_n(G)$ to $\mathbb{V}_{n+1}(G)$. By the above observation, at no point in the construction of $\mathbb{V}_\omega(G)$ a new member is added to an old member, which shows that extensionality is preserved in the limit. It is also easy to see that every vertex of $\mathbb{V}_\omega(G)$ has finite in-degree. To verify that $\mathbb{V}_\omega(G)$ is acyclic it suffices to check that each finite approximation $\mathbb{V}_n(G)$ is acyclic, so we shall verify that it is impossible for $\mathbb{V}_{n+1}(G)$ to have a cycle and for $\mathbb{V}_n(G)$ to be acyclic. So suppose there is a cycle $\langle s_i : 1 \leq i \leq k \rangle$ in $\mathbb{V}_{n+1}(G)$, and $\mathbb{V}_n(G)$ is acyclic. Then by Observation 3.4.1, for some i ,

$$s_i \in V_{n+1}(G) \setminus V_n(G).$$

This implies that s_i is a member of the deficiency set of $\mathbb{V}_n(G)$, and so $s_i \subseteq V_n(G)$. But for $j = i + 1 \pmod k$, we have $s_i E_{n+1} s_j$. This contradicts the definition of E_{n+1} , thereby completing our verification that G is acyclic. This completes the proof. $\square$

Remark 3.5. The proof of Theorem 3.4 makes it clear that G is end extended by $\mathbb{V}_\omega(G)$, and every element of $\mathbb{V}_\omega(G)$ is first order definable in the structure $(\mathbb{V}_\omega(G), c)_{c \in X}$.

Example 3.6.

(1) For every $S \subseteq V_\omega$, $\mathbb{V}_\omega(S, \in) = (V_\omega, \in)$.

(2) Let $G_\omega := (\omega, \{\langle n+1, n \rangle : n \in \omega\})$. $\mathbb{V}_\omega(G_\omega)$ is our first concrete example of a nonstandard ω -model of $\mathbf{ZF}_{\text{fin}}$.

Corollary 3.7. $\mathbf{ZF}_{\text{fin}}$ has nonstandard recursive ω -models.

Proof: Let us call a digraph $G = (\omega, E)$ on the set of natural numbers to be *highly recursive* if both the edge-set E and the map

$$n \mapsto n_E$$

are recursive. The constructive nature of the proof of Theorem 3.4 makes it evident that if $G = (\omega, E)$ is highly recursive digraph, then there is recursive $R \subseteq \omega^2$ such that $(\omega, R) \cong \mathbb{V}_\omega(G)$. Since the digraph G_ω of part (2) of Example 3.6 is easily seen to be highly recursive, we may conclude that there is a recursive $F \subseteq \omega^2$ such that $(\omega, F) \cong \mathbb{V}_\omega(G)$. $\square$

Remark 3.8. Note that the model (ω, F) constructed in the proof of Theorem 3.4 is itself highly recursive. Generally speaking, if $\mathfrak{M}$ is a highly recursive ω -model of $\mathbf{ZF}_{\text{fin}}$, then the set of natural numbers $\mathbb{N}^{\mathfrak{M}}$ is also recursive. However, it is easy to construct a recursive ω -model of $\mathbf{ZF}_{\text{fin}}$ in which the set of natural numbers $\mathbb{N}^{\mathfrak{M}}$ is not recursive. With more effort, one can even build a recursive ω -model of $\mathbf{ZF}_{\text{fin}}$ that is not isomorphic to a recursive model with a recursive set of natural numbers.

Corollary 3.9. $\mathbf{ZF}_{\text{fin}}$ has ω -models in every infinite cardinality.

Proof: For any infinite cardinal κ , and any digraph $G = (X, E)$, let $\kappa \times G := (\kappa \times X, F)$, where

$$\langle \alpha, x \rangle F \langle \beta, y \rangle \Leftrightarrow (\alpha = \beta) \wedge xEy.$$

It is easy to see that if G is an extensional, acyclic digraph, all of whose vertices have finite in-degree, then $\kappa \times G$ shares the same features. This makes it evident that if G_ω is as in part (2) of Example 3.6, then $\mathbb{V}_\omega(\kappa \times G_\omega)$ is an ω -model of $\mathbf{ZF}_{\text{fin}}$ of cardinality κ . $\square$

Remark 3.10. Corollary 3.9 shows that in contrast with $\mathbf{ZF}_{\text{fin}} + \mathbf{TC}$, within $\mathbf{ZF}_{\text{fin}}$ there is no definable bijection between the universe and the set of natural numbers.

4. MODELS WITH SPECIAL PROPERTIES

In this section we refine the method of the previous section in order to construct families of ω -models of $\mathbf{ZF}_{\text{fin}}$ with a variety of additional structural features, e.g., having a prescribed automorphism group, or being pointwise definable. The central result of this section is Theorem 4.1 below which shows that one can canonically code any prescribed *simple* graph (A, F) (where F is a set of 2-element subsets of A) into a

model of $\mathbf{ZF}_{\text{fin}}$, thereby yielding a great deal of control over the resulting ω -models of $\mathbf{ZF}_{\text{fin}}$.

Definition 4.1. In what follows, every digraph $G = (X, E)$ we consider will be such that $\omega \subseteq X$; and for $n \in \omega$, $n_E = \{0, 1, \dots, n-1\}$. In particular, $0_E = \emptyset$.

(a) Let $\varphi_1(x)$ be the formula in the language of set theory that expresses

“there is a sequence $\langle x_n : n < \omega \rangle$ with $x = x_0$ such that $x_n = \{n, x_{n+1}\}$ for all $n \in \omega$ ”.

At first sight, this seems to require an existential quantifier over infinite sequences (which is not possible in $\mathbf{ZF}_{\text{fin}}$), but this problem can be circumvented by writing $\varphi_1(x)$ as the sentence that expresses the equivalent statement

“for every positive $n \in \omega$, there is a sequence $s_n = \langle x_i : i < n \rangle$ with $x_0 = x$ such that for all $i < n-1$, $x_i = \{i, x_{i+1}\}$ ”.

(b) Using the above circumlocution, let $\theta(x, y, e)$ be the formula in the language of set theory that expresses

“ $x \neq y \wedge \varphi_1(x) \wedge \varphi_1(y)$, with corresponding sequences $\langle x_n : n \in \omega \rangle$, and $\langle y_n : n \in \omega \rangle$, and there is a sequence $\langle e_n : n \in \omega \rangle$ with $e = e_0$ such that $e_n = \{e_{n+1}, x_n, y_n\}$ for all $n \in \omega$ ”.

Then let $\varphi_2(x, y) := \exists e \theta(x, y, e)$.

Theorem 4.2. For every simple graph (A, F) there is an ω -model $\mathfrak{M}$ of $\mathbf{ZF}_{\text{fin}}$ whose universe contains A and which satisfies the following two conditions:

- (1) For all elements x and y of $\mathfrak{M}$, $\mathfrak{M} \models \varphi_2(x, y)$ iff $\{x, y\} \in F$;
- (2) Every element of $\mathfrak{M}$ is definable in $(\mathfrak{M}, c)_{c \in A}$.

Proof. Let

$$X = \omega \cup ((A \cup F) \times \omega).$$

For elements of $X \setminus \omega$, we write z_n instead of $\langle z, n \rangle$. This notation should be suggestive of how these elements are used in the definitions of $\varphi_1(x, y)$ and $\theta(x, y, e)$. Let

$$\begin{aligned} E = & \{ \langle m, n \rangle : m < n \in \omega \} \cup \\ & \{ \langle n, x_n \rangle : n \in \omega, x \in A \} \cup \\ & \{ \langle x_{n+1}, x_n \rangle : n \in \omega, x \in A \} \cup \\ & \{ \langle e_{n+1}, e_n \rangle : n \in \omega, e \in F \} \cup \\ & \{ \langle x_n, e_n \rangle : n \in \omega, x \in e \in F \}. \end{aligned}$$

Moreover, in order to arrange $A \subseteq X$ we can identify elements of the form $\langle x, 0 \rangle$ with x when $x \in A$. This gives $G = (X, E)$, which is an extensional, acyclic digraph all of whose vertices have finite in-degree. Let $\mathfrak{M} := \mathbb{V}_\omega(G)$. By Theorem 3.4, $\mathfrak{M}$ is an ω -model of $\mathbf{ZF}_{\text{fin}}$. It is clear that if $\{x, y\} \in F$, then $\mathfrak{M} \models \varphi_2(x, y)$. To see that the converse also holds it suffices to check that if $\mathfrak{M} \models \varphi_1(x)$, then $x \in A$, which in turns amounts to verifying that $V_\omega(G) \setminus X$ does not contain an element that satisfies φ_1 . This rests on the crucial observation that for *any* graph G , if the transitive closure $\tau(v)$ of an element $v \in V_\omega(G)$ is infinite, then a tail of $\tau(v)$ lies in G . The rest is easy and follows from Remark 3.5. $\square$

Note that part (1) of Theorem 4.2 implies that any automorphism f of $\mathfrak{M}$ induces an automorphism $\bar{f}$ of (A, F) , and part (2) of Theorem 4.2 shows that f is the unique extension of $\bar{f}$ to an automorphism of $\mathfrak{M}$. Since the map $f \mapsto \bar{f}$ is readily seen to be a group homomorphism between the automorphism groups of $\mathfrak{M}$ and (A, F) , this shows that $\mathfrak{M}$ and (A, F) have isomorphic automorphism groups, thereby allowing us to couple Theorem 4.2 with relevant theorems of graph theory to obtain the following results.

Corollary 4.3. *Every group can be realized as the automorphism group of an ω -model of $\mathbf{ZF}_{\text{fin}}$.*¹⁰

Proof. A classical theorem of Frucht [Fr] shows that every group can be realized as the automorphism group of a graph.¹¹ $\square$

Corollary 4.4. *For every infinite cardinal κ there are 2^κ nonisomorphic rigid ω -models of $\mathbf{ZF}_{\text{fin}}$ of cardinality κ .*

Proof. It is well-known that there are 2^κ nonisomorphic rigid simple graphs of cardinality κ . One way to see this is to first show that there are 2^κ nonisomorphic rigid linear orders of cardinality κ by the following construction: for each $S \subseteq \kappa$ let $\mathbb{L}_S$ be the linear order obtained by inserting a copy of ω^* (the order-type of negative integers) between α and $\alpha + 1$ for each $\alpha \in S$. Since $\mathbb{L}_S$ and $\mathbb{L}_{S'}$ are nonisomorphic for $S \neq S'$ this completes the argument since every linear order can be coded into a simple graph with the same automorphism group: given a linear order $(L, <_L)$, let $[L]^2$ be the set of all 2-element subsets of L , and for each $s \in [L]^2$, introduce distinct vertices $\{a_s, b_s\}$, and consider

¹⁰In contrast, the class of groups that can arise as the automorphism groups of models of $\mathbf{ZF}_{\text{fin}} + \text{TC}$ are precisely the right-orderable groups. This is a consequence of coupling the bi-interpretability of models of $\mathbf{ZF}_{\text{fin}} + \text{TC}$ and PA with a key result [KS, Theorem 5.4.4] about automorphisms of models of PA .

¹¹See [Lo] for an exposition of Frucht's theorem and its generalizations.

the simple graph (A, F) , where

$$A = L \cup \{a_s : s \in S\} \cup \{b_s : s \in S\},$$

and

$$\begin{aligned} F = & \{\{x, a_s\} : x \in s \in [L]^2\} \cup \\ & \{\{a_s, b_s\} : s \in [L]^2\} \cup \\ & \{\{x, b_s\} : s = \{x, y\}, \text{ and } x <_L y\}. \end{aligned}$$

□

Corollary 4.5. *For every infinite cardinal κ there is a family $\mathcal{M}$ of cardinality 2^κ of ω -models of $\mathbf{ZF}_{\text{fin}}$ of power κ such that for any distinct $\mathfrak{M}_1$ and $\mathfrak{M}_2$ in $\mathcal{M}$, there is no elementary embedding from $\mathfrak{M}_1$ into $\mathfrak{M}_2$.*

Proof. It is well-known that there is a family $\mathcal{G}$ of power 2^κ of simple graphs of cardinality κ such that for distinct G_1 and G_2 in $\mathcal{G}$, there is no embedding from G_1 into G_2 .¹² □

The next Corollary is motivated by the following observations: (1) $(V_\omega, \in)$ is a pointwise definable model, and (2) The Gödel-Rosser incompleteness theorem is powerless in giving any information on the number of complete extensions of $\mathbf{ZF}_{\text{fin}}$ that possess an ω -model.¹³

Corollary 4.6. *There are $2^{\aleph_0}$ pointwise definable ω -models of $\mathbf{ZF}_{\text{fin}}$. Consequently there are $2^{\aleph_0}$ complete extensions of $\mathbf{ZF}_{\text{fin}}$ that possess ω -models.*

Proof. In light of Theorem 4.2 and Remark 3.5, it suffices to show that there are $2^{\aleph_0}$ nonisomorphic pointwise definable simple graphs. Consider the simple graph $\mathcal{A} = (A, F)$ with

$$\begin{aligned} A = & \{x_n : n \in \omega\} \cup \\ & \{y_{nm} : m < n \in \omega\} \cup \\ & \{z_{nms} : s < m < n \in \omega\}. \end{aligned}$$

¹²Indeed, by a theorem of Nešetřil and Pultr [NP] one can strengthen this result by replacing “embedding” to “homomorphism”. We should also point out that by a result of Perminov [P] for each infinite cardinal κ , there are 2^κ nonisomorphic rigid graphs none of which can be embedded in to any of the others. This shows that Corollaries 4.5 and 4.6 can be dovetailed.

¹³The Gödel-Rosser theorem can be fine-tuned to show the essential undecidability of consistent first order theories that interpret Robinson’s Q [HP, Theorem 2.10, Chap. III], thereby showing that such theories have continuum-many consistent completions (indeed, the result continues to hold with Q replaced by the weaker system R).

(where the indicated members of A are all distinct), and

$$F = \{\{x_n, y_{nm}\} : m < n \in \omega\} \cup \{\{y_{nms}, z_{nms}\} : s < m < n \in \omega\}.$$

For each $S \subseteq \omega$, let $\mathcal{A}_S$ be the subgraph of $\mathcal{A}$ induced by the *transitive closure* of $\{x_n : n \in S\}$. It is routine to verify that each $\mathcal{A}_S$ is pointwise definable; and for distinct S and S' , $\mathcal{A}_S$ and $\mathcal{A}_{S'}$ are nonisomorphic. $\square$

The next corollary is an immediate consequence of coupling Theorem 4.2 with a classical construction [H, Theorem 5.5.1] that shows that for every structure $\mathfrak{A}$ in a *finite* signature there is a graph $G_{\mathfrak{A}}$ that interprets an isomorphic copy of $\mathfrak{A}$.

Corollary 4.7. *For every structure $\mathfrak{A}$ in a finite signature there is an ω -model $\mathfrak{M}$ of $\mathbf{ZF}_{\text{fin}}$ such that $\mathfrak{M}$ interprets an isomorphic copy of $\mathfrak{A}$.*

It is easy to see that $\mathbb{V}_\omega$ is the only ω -model of $\mathbf{ZF}_{\text{fin}} + \mathbf{TC}$ up to isomorphism. As it turns out, there are many other “categorical” ω -models of finite set theory. For example, if G happens to be an *arithmetical* extensional, acyclic graph, all of whose vertices have finite in-degree, and G has a vertex v such that every vertex of G is in the transitive closure of v , then $\mathbb{V}_\omega(G)$ is the only ω -model of $\mathbf{ZF}_{\text{fin}} + \varphi_G$ up to isomorphism, where φ_G is the sentence that expresses

$$“\exists s (V = V_\omega((\tau(s), \in)) \text{ and } (\tau(s), \in) \cong G)”.$$

This observation can be used to show that *there are precisely $\aleph_0$ finite extensions of $\mathbf{ZF}_{\text{fin}}$ that have exactly one ω -model up to isomorphism*, which in turn shows that there are infinitely many completions of $\mathbf{ZF}_{\text{fin}}$ that have ω -models. These considerations provide a context for the next theorem.

Theorem 4.8. *There are $2^{\aleph_0}$ completions of*

$$\mathbf{ZF}_{\text{fin}} + \exists s V = V_\omega((\tau(s), \in))$$

that possess a unique ω -model up to isomorphism.

Proof: Consider any $Y \subseteq \omega$. Define $f : \omega \rightarrow \omega$ so that $f(0) = 0$ and

$$f(n+1) = \begin{cases} f(n) + 1 & \text{if } n \in Y \\ f(n) + 2 & \text{if } n \notin Y \end{cases}$$

Let $G = (X, E)$ be such that

$$X = \omega \cup \{x_n : n \in \omega\},$$

and

$$\begin{aligned} E = & \{ \langle m, n \rangle : m < n \in \omega \} \cup \\ & \{ \langle x_{n+1}, x_n \rangle : n \in \omega \} \cup \\ & \{ \langle f(n), x_n \rangle : n \in \omega \}. \end{aligned}$$

Then G is an extensional, acyclic graph, all of whose vertices have finite in-degree; and $\mathbb{V}_\omega(G)$ is the only ω -model of its own theory up to isomorphism. $\square$

5. OPEN QUESTIONS AND CONCLUDING REMARKS

Question 5.1. *For infinite cardinals κ and λ , let $\mathcal{M}(\kappa, \lambda)$ be the class of models $\mathfrak{M}$ of $\mathbf{ZF}_{\text{fin}}$ such that the cardinality of $\mathfrak{M}$ is κ , and the cardinality of $\mathbb{N}^{\mathfrak{M}}$ is λ .*

(a) *Is there a first order scheme Γ_1 in the language of set theory such that $\text{Th}(\mathcal{M}(\omega_1, \omega)) = \mathbf{ZF}_{\text{fin}} + \Gamma_1$?*

(b) *Is there a first order scheme Γ_2 in the language of set theory such that $\text{Th}(\mathcal{M}(\beth_\omega, \omega)) = \mathbf{ZF}_{\text{fin}} + \Gamma_2$?*

Question 5.1 is motivated by classical two-cardinal theorems of Model Theory¹⁴, which show that the answers to both parts of Question 5.1 are positive if “first order scheme” is weakened to “recursively enumerable set of first order sentences”. The aforementioned two-cardinal theorems also show that (1) for $\kappa > \lambda \geq \omega$, $\text{Th}(\mathcal{M}(\kappa, \lambda)) = \text{Th}(\mathcal{M}(\omega_1, \omega))$, and (2) for all $\kappa \geq \beth_\omega$, $\text{Th}(\mathcal{M}(\kappa, \omega)) = \text{Th}(\mathcal{M}(\beth_\omega, \omega))$.

Question 5.2. *Let $\mathcal{L}$ be a language extending $\{\in\}$, and let $\mathbf{ZF}_{\text{fin}}(\mathcal{L})$ be the natural extension of $\mathbf{ZF}_{\text{fin}}$ in which the replacement scheme is allowed to use $\mathcal{L}$ -formulae (note that any $\mathcal{L}$ -expansion of an ω -model of $\mathbf{ZF}_{\text{fin}}$ satisfies $\mathbf{ZF}_{\text{fin}}(\mathcal{L})$).*

(a) *Let $f(x)$ be a unary function symbol, and let $\mathbf{GC}_f$ be the global choice axiom*

$$\mathbf{GC}_f := \forall x (x \neq \emptyset \rightarrow f(x) \in x).$$

Is $\mathbf{ZF}_{\text{fin}}(\{\in, f\}) + \mathbf{GC}_f$ a conservative extension of $\mathbf{ZF}_{\text{fin}}$?

(b) *More generally, let $\mathbf{SF}_f$ be the Skolem function scheme consisting of the universal closure of formulas of the form (where f is allowed to occur in φ)*

$$\forall x (\exists y \varphi(x, y) \rightarrow \varphi(x, f(x))).$$

Is $\mathbf{ZF}_{\text{fin}}(\{\in, f\}) + \mathbf{SF}_f$ a conservative extension of $\mathbf{ZF}_{\text{fin}}$?

¹⁴See [CK, Theorems 3.2.12. and 7.2.2], and [S-1].

Question 5.2 is motivated by the well-known forcing argument [F] establishing the conservativity of $\text{ZF}(\{\in, f\}) + \text{GC}_f$ over ZFC . We suspect that this forcing argument can be adapted to yield a positive answer to both parts of Question 5.2. Note that both parts of Question 5.2 have a positive answer if TC is added to ZF_{fin} since a parameter-free definable well-ordering of the universe is available in all models of $\text{ZF}_{\text{fin}} + \text{TC}$. It is worth pointing out that within $\text{ZF}(\{\in, f\}) + \text{GC}_f$ one can define a function $\bar{f}(x)$ such that $\text{ZF}(\{\in, f\}) + \text{GC}_f \vdash \text{SF}_{\bar{f}}$. This follows from the fact that Zermelo's proof of the well-ordering theorem within ZFC can be adapted to show that within $\text{ZF}(\{\in, f\}) + \text{GC}_f$ one can define a global well-ordering of the universe [Le, Prop. 4.1].

Let $\mathfrak{A} \equiv_n^{\text{Levy}} \mathfrak{B}$ abbreviate the assertion that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same Σ_n^{Levy} formulas in the usual Levy hierarchy of formulas of set theory (in which only unbounded quantification is significant).

Theorem 5.3. *If $\mathfrak{A}$ and $\mathfrak{B}$ are ω -models of ZF_{fin} , then $\mathfrak{A} \equiv_1^{\text{Levy}} \mathfrak{B}$.*

Proof: We use a variant of the Ehrenfeucht-Fraïssé game adapted to this context, in which once the first player (spoiler) and second player (duplicator) have made their first moves by choosing elements from each of the two structures, the players are obliged to select only members of elements that have been chosen already (by either party). We wish to show that for any particular length n of the play ($n > 0$), the duplicator has a winning strategy. Assume, without loss of generality, that the spoiler chooses $a_1 \in A$. The duplicator responds by choosing $b_1 \in B$ with the key property that there is function f such that

$$f : (\tau(n, a), \in)^{\mathfrak{A}} \xrightarrow{\cong} (\tau(n, b), \in)^{\mathfrak{B}}.$$

(it is easy to see, using the fact that $\mathfrak{A}$ and $\mathfrak{B}$ are ω -models of ZF_{fin} , that the duplicator can always do this). From that point on, once the spoiler picks any element c from either structure the duplicator responds with $f(c)$ or $f^{-1}(c)$ depending on whether c is in the domain or co-domain of f . $\square$

Note that TC is Π_2^{Levy} (since $\text{TC}(x)$ is Σ_1^{Levy}) relative to ZF_{fin} , so in the conclusion of Theorem 5.3, $\equiv_1^{\text{Levy}}$ cannot be replaced by $\equiv_2^{\text{Levy}}$ since $\mathfrak{A}$ can be chosen to be $(V_\omega, \in)$ and $\mathfrak{B}$ can be chosen to be a nonstandard ω -model of ZF_{fin} . These considerations motivate the next question.

Question 5.4. *Is it true that for each $n \geq 1$ there are ω -models $\mathfrak{A}$ and $\mathfrak{B}$ of ZF_{fin} that are $\equiv_n^{\text{Levy}}$ equivalent but not $\equiv_{n+1}^{\text{Levy}}$ equivalent?*

A natural approach to answering Question 5.4 would be to apply Theorem 3.4 to graphs whose first order theories are distinct, yet agree

up to some specified level of complexity. Such an approach should be able to deal with the problem *that the first order theory of G does not, in general, determine the first order theory of $\mathbb{V}_\omega(G)$* . To see this, let G_1 be the graph $(\mathbb{Z}, E)$, where $\mathbb{Z}$ is the set of integers, and $E = \{\langle n+1, n \rangle : n \in \mathbb{Z}\}$; and let G_2 be the disjoint sum of two copies of G_1 . Then G_1 and G_2 are elementarily equivalent, but $\mathbb{V}_\omega(G_1)$ and $\mathbb{V}_\omega(G_2)$ are not elementarily equivalent, indeed, both $\mathbb{V}_\omega(G_1)$ and $\mathbb{V}_\omega(G_2)$ are the unique ω -models of their own first order theories up to isomorphism.

REFERENCES

- [BF] S. Baratella and R. Ferro, *A theory of sets with the negation of the axiom of infinity*, **Mathematical Logic Quarterly** vol. 39 (1993), pp. 338–352.
- [B-1] P. Bernays, *A system of axiomatic set theory (Part II)*, **J. Symbolic Logic** vol. 6, (1941), pp. 1–17.
- [B-2] —————, *A system of axiomatic set theory (Part VII)*, **J. Symbolic Logic** vol. 19, (1954), pp. 81–96.
- [CK] C. C. Chang and H. J. Keisler, **Model Theory**, North-Holland, Amsterdam 1973.
- [F] U. Felgner, *Comparisons of the axioms of local and global choice*, **Fund. Math.** vol. 71 (1971), pp. 43–62.
- [Fr] R. Frucht, *Herstellung von Graphen mit vorgegebener abstrakter Gruppe*, **Compositio Math.** vol. 6 (1939), pp. 239–250.
- [HP] P. Hájek and P. Pudlák, **Metamathematics of First-order Arithmetic**, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998.
- [H] W. Hodges, **Model theory**, Encyclopedia of Mathematics and its Applications, vol. 42, Cambridge University Press, Cambridge, 1993.
- [Ka] R. Kaye, *Tennenbaum’s theorem for models of arithmetic*, this volume.
- [KW] R. Kaye and T. Wang, *On interpretations of arithmetic and set theory*, **Notre Dame Journal of Formal Logic**, vol. 48, Number 4 (2007), pp. 497–510.
- [Ke] H.J. Keisler, **Model theory for Infinitary Logic**, Studies in Logic and the Foundations of Mathematics, vol. 62. North-Holland Publishing Co., Amsterdam-London, 1971.
- [KS] R. Kossak, and J.H. Schmerl, **The Structure of Models of Peano Arithmetic**, Oxford Logic Guides, vol. 50. Oxford Science Publications, The Clarendon Press, Oxford University Press, Oxford, 2006.
- [Kr] Kreisel, G. *Note on arithmetic models for consistent formulae of the predicate calculus. II. Actes du XIème Congrès International de Philosophie*, Bruxelles, 20–26 Août 1953, vol. XIV, pp. 39–49. North-Holland Publishing Co., Amsterdam; Éditions E. Nauwelaerts, Louvain, 1953.
- [Le] A. Levy, **Basic set theory**, Dover Publications, Inc., Mineola, NY, 2002.
- [Lo] L. Lovász, **Combinatorial Problems and Exercises** (Corrected reprint of the 1993 second edition), AMS Chelsea Publishing, Providence, RI, 2007.
- [MZ] A. Mancini and D. Zambella, *A note on recursive models of set theories*, **Notre Dame J. Formal Logic** vol. 42 (2001), no. 2, pp. 109–115.

- [Mo] Mostowski, A. *On a system of axioms which has no recursively enumerable arithmetic model*, **Fundamenta Mathematicae** vol. 40, (1953). 56–61.
- [M] S. Mohsenipour, *A recursive nonstandard model for open induction with GCD property and cofinal primes*, in **Logic in Tehran**, pp. 227–238, Lect. Notes Log., vol. 26, Assoc. Symbol. Logic, La Jolla, CA, 2006.
- [NP] J. Nešetřil and A. Pultr, *A note on homomorphism-independent families*, **Combinatorics** (Prague, 1998). **Discrete Math.** vol. 235 (2001), no. 1-3, pp. 327–334.
- [P] E. A. Perminov, *The number of rigid graphs that are mutually nonimbeddable into one another*, **Izv. Vyssh. Uchebn. Zaved. Mat.** (1985), no. 8, vol. 86, pp. 77–78.
- [Ra] M. Rabin, *On recursively enumerable and arithmetic models of set theory*, **J. Symbolic Logic** vol. 23 (1958), no. 4, pp. 408–416.
- [Ri] L. Rieger, *A contribution to Gödel's axiomatic set theory*, **Czechoslovak Math. J.** vol. 7 (1957), pp. 323–357.
- [S-1] J. H. Schmerl, *An axiomatization for a class of two-cardinal models*, **J. Symbolic Logic** vol. 42 (1977), no. 2, pp. 174–178.
- [S-2] ———, *Tennenbaum's theorem and recursive reducts*, this volume.
- [Vi] A. Visser, *Categories of theories and interpretations*, in **Logic in Tehran** pp. 284–341, Lect. Notes Log., vol. 26, Assoc. Symbol. Logic, La Jolla, CA, 2006.
- [Vo-1] P. Vopěnka, *Axiome der Theorie endlicher Mengen*, **Cas. Pest. Mat.** vol. 89 (1964), pp. 312–317.
- [Vo-2] ———, **Mathematics in the Alternative Set Theory**, Teubner-Verlag, Leipzig, 1979.

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